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On the least values of L_p-norms for the Kontorovich–Lebedev transform and its convolution

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Abstract

We establish analogs of the Hausdorff–Young and Riesz–Kolmogorov inequalities and the norm estimates for the Kontorovich–Lebedev transformation and the corresponding convolution. These classical inequalities are related to the norms of the Fourier convolution and the Hilbert transform in L_p spaces, $1 \leqslant p \leqslant \infty$. Boundedness properties of the Kontorovich–Lebedev transform and its convolution operator are investigated. In certain cases the least values of the norm constants are evaluated. Finally, it is conjectured that the norm of the Kontorovich–Lebedev operator $K_{i\tau} : L_p(\mathbb{R}_+; x\,dx) \to L_p(\mathbb{R}_+; x \sinh \pi x\,dx),\ 2 \leqslant p \leqslant \infty$

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x) f(x)\,dx, \quad \tau \in \mathbb{R}_+$$

is equal to $\frac{\pi}{2^{1-\frac{1}{p}}}$. It confirms, for instance, by the known Plancherel-type theorem for this transform when $p = 2$.
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1. Introduction and preliminary results

Let f, h be complex-valued measurable functions defined on $\mathbb{R}_+ = (0, \infty)$. The purpose of this paper is to obtain an analog of the Hausdorff–Young inequality [2] and the norm estimates for the following convolution operator (cf. [6,7,11])

$$(f * h)(x) = \frac{1}{2x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x \frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} f(u)h(y)\, du\, dy, \quad x > 0. \tag{1.1}$$

Our aim is also to establish an analog of the Riesz–Kolmogorov inequality [3] for the Kontorovich–Lebedev transformation [5,7,8,11]

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x) f(x)\, dx, \quad \tau > 0, \tag{1.2}$$

where $K_{i\tau}(x)$ is the modified Bessel function of the second kind [1] with respect to the pure imaginary index $\nu = i\tau$. We will consider these operators in appropriate Lebesgue spaces. In particular, the convolution operator (1.1) is well defined in the Banach ring $L^\alpha(\mathbb{R}_+) \equiv L_1(\mathbb{R}_+; K_\alpha(x)\, dx)$, $\alpha \in \mathbb{R}$ (see [11,7]), i.e. the space of all summable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with respect to the measure $K_\alpha(x)\, dx$ for which

$$||f||_{L^\alpha(\mathbb{R}_+)} = \int_0^\infty |f(x)| K_\alpha(x)\, dx \tag{1.3}$$

is finite. Generally, the modified Bessel function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2) u = 0 \tag{1.4}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. It has the asymptotic behaviour (cf. [1, relations (9.6.8), (9.6.9), (9.7.2)])

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \quad z \to \infty \tag{1.5}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\, \nu|}\right), \quad z \to 0, \tag{1.6}$$

$$K_0(z) = -\log z + O(1), \quad z \to 0. \tag{1.7}$$

Moreover, it can be defined by the following integral representations [5, (6-1-2), 4, vol. I, relation (2.4.18.4)]

$$K_\nu(x) = \int_0^\infty e^{-x \cosh u} \cosh \nu u\, du, \quad x > 0, \tag{1.8}$$

$$K_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-t - \frac{x^2}{4t}} t^{-\nu-1}\, dt, \quad x > 0. \tag{1.9}$$

Hence, we easily find that $K_\nu(x)$ is a real-valued positive function when $\nu \in \mathbb{R}$ and an even function with respect to the index ν. Moreover, it satisfies the following inequalities:

$$|K_\nu(x)| \leqslant K_{\mathrm{Re}\,\nu}(x), \; x > 0, \tag{1.10}$$

$$x^{|\mathrm{Re}\,\nu|}|K_\nu(x)| \leqslant 2^{|\mathrm{Re}\,\nu|-1}\Gamma(|\mathrm{Re}\,\nu|), \quad x > 0, \quad \mathrm{Re}\,\nu \neq 0, \tag{1.11}$$

where $\Gamma(z)$ is Euler's gamma-function [1]. Applying the Hölder inequality to the integral (1.3) and invoking asymptotic formulas (1.5)–(1.7) with the latter inequalities (1.10), (1.11) for the weighted function $K_\alpha(x)$ it is not difficult to establish the following embeddings:

$$L^\alpha(\mathbb{R}_+) \equiv L^{-\alpha}(\mathbb{R}_+), \; L^\alpha(\mathbb{R}_+) \subseteq L^\beta(\mathbb{R}_+), \; |\alpha| \geqslant |\beta| \geqslant 0, \alpha, \beta \in \mathbb{R}, \tag{1.12}$$

$$L^\alpha(\mathbb{R}) \supset L_p(\mathbb{R}_+; x\,dx), \; 2 < p \leqslant \infty, \; |\alpha| < 1 - \frac{2}{p}, \tag{1.13}$$

where $L_p(\mathbb{R}_+; x\,dx)$ is a weighted Banach space with the norm

$$||f||_{L_p(\mathbb{R}_+;x\,dx)} = \left(\int_0^\infty |f(x)|^p x\,dx\right)^{1/p}, \quad 1 \leqslant p < \infty, \tag{1.14}$$

$$||f||_{L_\infty(\mathbb{R}_+;x\,dx)} = \mathrm{ess\ sup}_{x\in\mathbb{R}_+} |f(x)|. \tag{1.15}$$

Our goal in this paper is to study the boundedness properties in $L_p(\mathbb{R}_+; x\,dx)$ of the convolution operator (1.1) when one of the functions, say h, is fixed and belongs to the space $L^\alpha(\mathbb{R}_+)$, where α depends on p. For this we will generalize on L_p-case the following Hausdorff–Young-type inequality:

$$||f * h||_{L_2(\mathbb{R}_+;x\,dx)} \leqslant ||f||_{L_2(\mathbb{R}_+;x\,dx)}||h||_{L^0(\mathbb{R}_+)}, \tag{1.16}$$

which is proved in [9,10]. Furthermore, for some values of p relations between norms of operators (1.1), (1.2) are confirmed by the Young-type inequality (see [10])

$$||f * h||_{L_{1/\mu}(\mathbb{R}_+;x\,dx)} \leqslant C_{\mu,\gamma,\beta}||f||_{L_{1/\beta}(\mathbb{R}_+;x\,dx)}||h||_{L_{1/\gamma}(\mathbb{R}_+;x\,dx)}, \tag{1.17}$$

where

$$C_{\mu,\gamma,\beta} = \left(\int_0^\infty x^{1-\frac{\gamma+\beta}{\mu}} K^{\frac{1-\gamma-\beta}{\mu}}_{\frac{\gamma-\beta}{1-\gamma-\beta}}(x) K_0^{\frac{\gamma+\beta}{\mu}}(x)\,dx\right)^\mu, \tag{1.18}$$

and $0 < \gamma, \beta < 1, \; \gamma + \beta \leqslant 1, \; \frac{\gamma+\beta+|\gamma-\beta|}{2} < \mu \leqslant 1$.

In the final section, we prove an analog of the Riesz–Kolmogorov theorem for the Kontorovich–Lebedev operator (1.2) when $2 \leqslant p \leqslant \infty$. Namely, we will prove the following inequality:

$$\int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^p\,d\tau \leqslant \frac{\pi^p}{2^{p-1}} \int_0^\infty x|f(x)|^p\,dx, \tag{1.19}$$

which is equivalent to

$$||K_{i\tau}[f]||_{L_p(\mathbb{R}_+;\tau \sinh \pi\tau\,d\tau)} \leqslant \frac{\pi}{2^{1-\frac{1}{p}}}||f||_{L_p(\mathbb{R}_+;x\,dx)}. \tag{1.20}$$

We note, that spaces $L_p(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$, $p \geqslant 1$ and $L_\infty(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$ are normed, respectively, by

$$||f||_{L_p(\mathbb{R}_+;\tau \sinh \pi\tau\, d\tau)} = \left(\int_0^\infty |f(\tau)|^p \tau \sinh \pi\tau\, d\tau\right)^{1/p}, \quad 1 \leqslant p < \infty, \tag{1.21}$$

$$||f||_{L_\infty(\mathbb{R}_+;\tau \sinh \pi\tau\, d\tau)} = \text{ess sup}_{\tau\in\mathbb{R}_+} |f(\tau)|. \tag{1.22}$$

We will conjecture that the norm of the Kontorovich–Lebedev operator (1.2) $||K_{i\tau}||$ is equal to $\frac{\pi}{2^{1-\frac{1}{p}}}$, where we define the norm as usual by

$$||K_{i\tau}|| = \sup_{||f||_{L_p(\mathbb{R}_+;x\,dx)}=1} ||K_{i\tau}[f]||_{L_p(\mathbb{R}_+;\tau \sinh \pi\tau\, d\tau)}. \tag{1.23}$$

As it is known, the product of the modified Bessel functions of the second kind of different arguments can be represented by the Macdonald formula [4, vol. II, relation (2.16.9.1)]

$$K_\nu(x)K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} K_\nu(u)\frac{du}{u}. \tag{1.24}$$

This is a key formula, which is used to prove the factorization property for the convolution (1.1) in terms of the Kontorovich–Lebedev transform (1.2) in the space $L^\alpha(\mathbb{R}_+)$ [9,11], namely

$$K_{i\tau}[f * h] = K_{i\tau}[f]K_{i\tau}[h], \tau \in \mathbb{R}_+, \tag{1.25}$$

where the integral (1.2) exists as a Lebesgue integral. It is also proved in [7,11] that the Kontorovich–Lebedev transform is a bounded operator from $L^\alpha(\mathbb{R}_+)$ into the space of bounded continuous functions on $\mathbb{R}_+$ vanishing at infinity. Furthermore, the convolution (1.1) of two functions $f, h \in L^\alpha(\mathbb{R}_+)$ exists as a Lebesgue integral and belongs to $L^\alpha(\mathbb{R}_+)$. It satisfies the Young-type inequality

$$||f * h||_{L^\alpha(\mathbb{R}_+)} \leqslant ||f||_{L^\alpha(\mathbb{R}_+)}||h||_{L^\alpha(\mathbb{R}_+)}. \tag{1.26}$$

However, it is not difficult to verify that in the case $f \in L_2(\mathbb{R}_+; x\,dx)$ integral (1.2) in general, does not exist in Lebesgue's sense (take, for instance

$$f(x) = \begin{cases} \frac{1}{x \log x} & \text{if } 0 < x \leqslant \frac{1}{2}, \\ 0 & \text{if } x > \frac{1}{2}, \end{cases}$$

and use asymptotic formula (1.6)). Thus we define it in the form

$$K_{i\tau}[f] = \lim_{N\to\infty}\int_{1/N}^N K_{i\tau}(x)f(x)\,dx, \tag{1.27}$$

where the limit is taken in the mean-square sense with respect to the norm of the space $L_2(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$. It has been proved (see [7]) that

$$K_{i\tau} : L_2(\mathbb{R}_+; x\,dx) \leftrightarrow L_2(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$$

is a bounded operator and forms an isometric isomorphism between these Hilbert spaces with the Parseval identity of the form

$$\int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 \, d\tau = \frac{\pi^2}{2} \int_0^\infty x|f(x)|^2 \, dx. \tag{1.28}$$

The two definitions (1.2) and (1.27) are equivalent, if $f \in L_2(\mathbb{R}_+; x\,dx) \cap L^\alpha(\mathbb{R}_+)$. The inverse operator in the latter case is given by the formula $f(x) = \lim_{N\to\infty} f_N(x)$, where

$$f_N(x) = \frac{2}{\pi^2} \int_0^N \tau \sinh \pi\tau \, \frac{K_{i\tau}(x)}{x} K_{i\tau}[f] \, d\tau, \tag{1.29}$$

and the convergence is in the mean-square sense with respect to the norm (1.14) of $L_2(\mathbb{R}_+; x\,dx)$. It can be written for almost all $x \in \mathbf{R}_+$ in the equivalent form

$$f(x) = \frac{2}{x\pi^2} \frac{d}{dx} \int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[f] \, dy \, d\tau. \tag{1.30}$$

2. Boundedness properties of the convolution operator

We generalize inequality (1.16) by proving the following:

Theorem 1. *Let* $1 < p \leqslant \infty$, $f \in L_p(\mathbb{R}_+; x\,dx)$ *and* $h \in L^{\frac{p-2}{p-1}}(\mathbb{R}_+)$. *Then convolution* (1.1) *exists as a Lebesgue integral for all* $x > 0$ *and belongs to the space* $L_p(\mathbb{R}_+; x\,dx)$. *Moreover, it satisfies the following inequality*:

$$||f * h||_{L_p(\mathbb{R}_+; x\,dx)} \leqslant ||f||_{L_p(\mathbb{R}_+; x\,dx)} ||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}. \tag{2.1}$$

Proof. Indeed, from Hölder's inequality we have for convolution (1.1) the estimate ($q^{-1} + p^{-1} = 1$)

$$|(f*h)(x)|^p \leqslant \frac{1}{(2x)^p} \left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} |h(y)| \frac{du\,dy}{u^{q/p}} \right)^{p/q}$$
$$\times \int_0^\infty \int_0^\infty u|f(u)|^p e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} |h(y)| \, du \, dy. \tag{2.2}$$

Hence we use the formula (see in [4, vol. I, relation (2.3.16.1)])

$$\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} \frac{du}{u^{q/p}} = 2\left(\frac{x^2y^2}{x^2+y^2}\right)^{(1-q/p)/2} K_{1-q/p}\left(\sqrt{x^2+y^2}\right) \tag{2.3}$$

and we prove that for all $x, y > 0$

$$\left(\frac{x^2y^2}{x^2+y^2}\right)^{(1-q/p)/2} K_{1-q/p}\left(\sqrt{x^2+y^2}\right) \leqslant x^{1-q/p} K_{1-q/p}(y). \tag{2.4}$$

Indeed, invoking the representation (1.9) we deduce

$$\left(\frac{x^2y^2}{x^2+y^2}\right)^{(1-q/p)/2} K_{1-q/p}\left(\sqrt{x^2+y^2}\right) = (xy)^{1-q/p}2^{q/p-2} \times \int_0^\infty e^{-t-\frac{x^2+y^2}{4t}} t^{-(1-q/p)-1}\,dt \leqslant (xy)^{1-q/p}2^{q/p-2}\int_0^\infty e^{-t-\frac{y^2}{4t}} t^{-(1-q/p)-1}\,dt = (xy)^{1-q/p}2^{q/p-1}\left(\frac{2}{y}\right)^{1-q/p} K_{1-q/p}(y) = x^{1-q/p}K_{1-q/p}(y).$$

Thus by taking relations (2.3), (2.4) we estimate the right-hand side of the inequality (2.2) as follows:

$$|(f*h)(x)|^p \leqslant \frac{1}{2x^2}\left(\int_0^\infty K_{1-q/p}\left(\sqrt{x^2+y^2}\right)|h(y)|\,dy\right)^{p/q} \times \int_0^\infty\int_0^\infty u|f(u)|^p e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} \times |h(y)|\,du\,dy \leqslant \frac{1}{2x^2}\left(\int_0^\infty K_{1-q/p}(y)|h(y)|\,dy\right)^{p/q} \times \int_0^\infty\int_0^\infty u|f(u)|^p e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} \times |h(y)|\,du\,dy. \tag{2.5}$$

Hence multiplying both sides of (2.5) by x we integrate with respect to $x \in \mathbb{R}_+$. Inverting the order of integration by the Fubini theorem we invoke again (2.2) and an elementary inequality $K_0\left(\sqrt{u^2+y^2}\right) \leqslant K_0(y)$ to obtain

$$\int_0^\infty x|(f*h)(x)|^p dx \leqslant \left(\int_0^\infty K_{1-q/p}(y)|h(y)|\,dy\right)^{p/q}\int_0^\infty\int_0^\infty uK_0\left(\sqrt{u^2+y^2}\right)|f(u)|^p \times |h(y)|\,du\,dy \leqslant \int_0^\infty u|f(u)|^p\,du\left(\int_0^\infty K_{1-q/p}(y)|h(y)|\,dy\right)^{p/q} \times \int_0^\infty K_0(y)|h(y)|\,dy. \tag{2.6}$$

Hence taking into account that $q = p/(p-1)$ we recall norms (1.3), (1.14) and write (2.6) in the form

$$||f*h||_{L_p(\mathbb{R}_+;x\,dx)} \leqslant ||f||_{L_p(\mathbb{R}_+;x\,dx)}||h||^{1/p}_{L^0(\mathbb{R}_+)}||h||^{1-1/p}_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}. \tag{2.7}$$

However, it is easily to verify from representation (1.8) that $K_0(y) \leqslant K_{1-q/p}(y)$. Hence from (2.7) (cf. embedding (1.12)) we arrive at the desired inequality (2.1). Theorem 1 is proved. □

Remark 1. Putting in (2.1) $p = 2$ we arrive at (1.16). When $p = \infty$ it takes the form

$$||f * h||_{L_\infty(\mathbb{R}_+;x\,dx)} \leqslant ||f||_{L_\infty(\mathbb{R}_+;x\,dx)} ||h||_{L^1(\mathbb{R}_+)}. \tag{2.8}$$

Another version of the Hausdorff–Young-type theorem for convolution (1.1) when f, h belong to the conjugate spaces (1.14) is given by

Theorem 2. *Let* $1 < p < \infty$, $h \in L_p(\mathbb{R}_+; x\,dx)$, $f \in L_q(\mathbb{R}_+; x\,dx)$, $q = p/(p-1)$. *Then convolution* (1.1) *exists as a Lebesgue integral for all* $x > 0$ *and belongs to the space* $L_r(\mathbb{R}_+; x^r\,dx)$ *with* $1 \leqslant r < \frac{pq}{2|p-q|}$. *Furthermore it satisfies the inequality*

$$||f * h||_{L_r(\mathbb{R}_+;x^r dx)} \leqslant C_{r,p,q} ||f||_{L_q(\mathbb{R}_+;x\,dx)} ||h||_{L_p(\mathbb{R}_+;x\,dx)}, \tag{2.9}$$

where

$$C_{r,p,q} = \left(\int_0^\infty \left[K_{1-q/p}^{1/q}(x) K_{1-p/q}^{1/p}(x) \right]^r dx \right)^{1/r}. \tag{2.10}$$

Proof. Again with Hölder's inequality and employing (2.3), (2.4) we majorize (1.1) as

$$\begin{aligned}
|(f * h)(x)| &\leqslant \frac{1}{2x} \left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} |h(y)|^p y \frac{du\,dy}{u^{p/q}} \right)^{1/p} \\
&\quad \times \left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} |f(u)|^q u \frac{du\,dy}{y^{q/p}} \right)^{1/q} \\
&= \frac{1}{x} \left(\int_0^\infty \left(\frac{x^2y^2}{x^2+y^2} \right)^{(1-p/q)/2} K_{1-p/q}\left(\sqrt{x^2+y^2}\right) |h(y)|^p y\,dy \right)^{1/p} \\
&\quad \times \left(\int_0^\infty \left(\frac{x^2u^2}{x^2+u^2} \right)^{(1-q/p)/2} K_{1-q/p}\left(\sqrt{x^2+u^2}\right) |f(u)|^q u\,du \right)^{1/q} \\
&\leqslant x^{-1} K_{1-q/p}^{1/q}(x) K_{1-p/q}^{1/p}(x) ||f||_{L_q(\mathbb{R}_+;x\,dx)} ||h||_{L_p(\mathbb{R}_+;x\,dx)}.
\end{aligned}$$

Hence we easily obtain inequality (2.9) with a constant given by the integral (2.10). Due to asymptotic formulas (1.5)–(1.7) we verify by straightforward calculations that it converges when $1 \leqslant r < \frac{pq}{2|p-q|}$. Theorem 2 is proved. □

Letting $p = q = 2,\ r = 1$ in (2.9) we invoke relation (2.16.2.1) from [4, vol. II] to calculate the corresponding value of the integral (2.10). As a result we deduce the inequality

$$||f * h||_{L_1(\mathbb{R}_+;x\,dx)} \leqslant \frac{\pi}{2} ||f||_{L_2(\mathbb{R}_+;x\,dx)} ||h||_{L_2(\mathbb{R}_+;x\,dx)}, \tag{2.11}$$

which is a particular case of (1.17) when $\mu = 1,\ \beta = \gamma = \frac{1}{2}$.

For a fixed function h we denote by

$$S_h(x,u) = \frac{1}{2x} \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} h(y)\,dy \tag{2.12}$$

the kernel of the following convolution operator:

$$(S_h f)(x) = \int_0^\infty S_h(x,u) f(u)\,du, \quad x > 0. \tag{2.13}$$

As a consequence of Theorem 1 we establish boundedness properties of the operator (2.13) in the space $L_p(\mathbb{R}_+; x\,dx),\ 1 < p \leqslant \infty$. Thus we have

Theorem 3. *Let* $1 < p \leqslant \infty$, *and* $h \in L^{\frac{p-2}{p-1}}(\mathbb{R}_+)$. *Then integral operator* (2.13) *is bounded in the space* $L_p(\mathbb{R}_+; x\,dx)$ *and its norm* $||S_h|| \leqslant ||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}$. *If, in turn,* $h(x)$ *is a positive function on* $\mathbb{R}_+$ *and* $p \in \left(1 + \frac{1}{\sqrt{3}}, \infty\right)$, *then* $||S_h|| = ||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}$.

Proof. The first part of the theorem easily follows from inequality (2.1). Let us show that if $h(x)$ is a positive function on $\mathbb{R}_+$ and $p \in \left(1 + \frac{1}{\sqrt{3}}, \infty\right)$, then the norm of the convolution operator (2.13) is equal to $||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}$. To do this we prove that the case $||S_h|| < ||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}$ is impossible when $p \in \left(1 + \frac{1}{\sqrt{3}}, \infty\right)$. Indeed, assuming that $||S_\varphi|| < 1$, where

$$\varphi(x) = \frac{h(x)}{||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}}, \quad x > 0$$

it follows immediately by virtue of the Banach theorem that the operator $A = I - S_\varphi$ (I denotes the identity operator) has an inverse in $L_p(\mathbb{R}_+; x\,dx),\ 1 < p < \infty$. Consequently, the adjoint operator $A^* = I - \hat{S}_\varphi$, where

$$(\hat{S}_\varphi f)(x) = \int_0^\infty S_\varphi(u,x) f(u)\,du, \quad x > 0, \tag{2.14}$$

has an inverse in $L_q(\mathbb{R}_+; x\,dx), q^{-1} + p^{-1} = 1$. However, we will show now that the operator A^* in $L_q(\mathbb{R}_+; x\,dx)$ cannot have an inverse since integral equation $A^* f = 0$ has a nontrivial solution $f(x) = K_{\frac{p-2}{p-1}}(x)$, which belongs to $L_q(\mathbb{R}_+; x\,dx)$ when $p \in$

$\left(1+\frac{1}{\sqrt{3}}, \infty\right)$. Indeed, employing the Macdonald formula (1.24) and the norm definition (1.3) of the space $L^{\alpha}(\mathbb{R}_+)$ we substitute formally in (2.14) the function $K_{\frac{p-2}{p-1}}(x)$ to obtain

$$\int_0^\infty S_\varphi(u,x) K_{\frac{p-2}{p-1}}(u)\, du = K_{\frac{p-2}{p-1}}(x)$$

for any positive $h(x) \in L^{\frac{p-2}{p-1}}(\mathbb{R}_+)$. This solution belongs to $L_q(\mathbb{R}_+; x\,dx)$, if the integral

$$\int_0^\infty K^q_{\frac{p-2}{p-1}}(x) x\, dx < \infty.$$

When $p = 2$ it evidently converges via asymptotic formulas (1.5), (1.7) for the modified Bessel function. For other values of p according to (1.5), (1.6) we see that the latter integral is convergent when $\frac{|p-2|p}{(p-1)^2} < 2$. This gives the condition $p \in \left(1+\frac{1}{\sqrt{3}}, \infty\right)$. So this fact contradicts our assumption above and we conclude that $||S_\varphi|| \geqslant 1$. But from the first part of the theorem it follows that $||S_\varphi|| \leqslant 1$. Thus we get $||S_\varphi|| = 1$ or $||S_h|| = ||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}$. Theorem 3 is proved. □

Let us consider two examples of convolution operator (2.13) (cf. [7,10]) with the corresponding kernels (2.12), which are calculated for concrete functions *h*. If, for instance, we put $h(x) \equiv 1$ then we calculate integral (2.12) by using representation (1.9) and we arrive at the following integral operator

$$(\mathcal{K}f)(x) = \int_0^\infty \frac{K_1\left(\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}} u f(u)\, du, \quad x > 0. \tag{2.15}$$

It is easily seen from (1.3), (1.5)–(1.7) that $h(x) \equiv 1$ belongs to the space $L^{\frac{p-2}{p-1}}(\mathbb{R}_+)$ if and only if $p \in \left(\frac{3}{2}, \infty\right)$. Consequently, appealing to Theorem 3 via relation (2.16.2.1) from [4, vol. II] we find that (2.15) is a bounded operator in $L_p(\mathbb{R}_+; x\,dx)$, $p \in \left(\frac{3}{2}, \infty\right)$ and has a least value of its norm when $p \in \left(1+\frac{1}{\sqrt{3}}, \infty\right)$, namely

$$||\mathcal{K}|| = \frac{\pi}{2\cosh\left(\frac{\pi}{2}\frac{p-2}{p-1}\right)}, \quad p \in \left(1+\frac{1}{\sqrt{3}}, \infty\right).$$

When, in turn, we put $h(x) = \frac{e^{-x}}{\sqrt{x}}$ then calculating the corresponding integral (2.12) and taking into account, that the modified Bessel function of the index $\frac{1}{2}$ reduces to

$$K_{1/2}(z) = e^{-z}\sqrt{\frac{\pi}{2z}}$$

(see [1, (9.6)]), we arrive at the Lebedev convolution operator (cf. [7,11])

$$(Lef)(x) = \sqrt{\frac{\pi}{2x}} \int_0^\infty \frac{e^{-x-u}}{x+u} \sqrt{u} f(u)\, du, \quad x > 0. \tag{2.16}$$

In this case $h(x) = \frac{e^{-x}}{\sqrt{x}}$ belongs to the space $L^{\frac{p-2}{p-1}}(\mathbb{R}_+)$ if and only if $p \in (\frac{5}{3}, 3)$. Moreover, using relation (2.16.6.4) from [4, vol. II, Theorem 3] we obtain that the Lebedev operator (2.16) is bounded in $L_p(\mathbb{R}_+; x\,dx)$, $p \in (\frac{5}{3}, 3)$ and we have

$$||Le|| = \pi\sqrt{\frac{\pi}{2}}\frac{1}{\cosh\left(\pi\frac{p-2}{p-1}\right)}, \quad p \in \left(\frac{5}{3}, 3\right).$$

3. Riesz–Kolmogorov-type theorem

In this final section, we study the Kontorovich–Lebedev transformation (1.2) as an operator, which maps the weighted space $L_p(\mathbb{R}_+; x\,dx)$ into the weighted space $L_p(\mathbb{R}_+; \tau \sinh \pi\tau\,d\tau)$ for $2 \leqslant p \leqslant \infty$. We will show that for $1 \leqslant p < 2$ this map in general, does not exist. Finally, we prove an analog of the Riesz–Kolmogorov theorem [3], which is known for the Hilbert transform in L_p. Namely, we will prove inequality (1.19) and consider least values of the norm constants (1.23) for the Kontorovich–Lebedev operator (1.2).

The case $p = 2$ has been studied in [7,9,10]. It forms an isometric isomorphism between the corresponding Hilbert spaces with the Parseval equality (1.28). It has also a relationship with convolution (1.1) by means of the following Parseval equality (cf. [9,10])

$$\int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]K_{i\tau}[h]|^2\,d\tau = \frac{\pi^2}{2}\int_0^\infty |(f * h)(x)|^2 x\,dx. \tag{3.1}$$

However, when $2 < p \leqslant \infty$ and $f(x) \in L_p(\mathbb{R}_+; x\,dx)$ it has been shown in [8] that integral (1.2) is understood as a Lebesgue integral. We will prove that the image of the Kontorovich–Lebedev operator belongs to $L_p(\mathbb{R}_+; \tau \sinh \pi\tau\,d\tau)$. It fails certainly when $1 \leqslant p < 2$. For instance, we take

$$f(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Then f clearly belongs to $L_p(\mathbb{R}_+; x\,dx)$, $1 \leqslant p < 2$. Nevertheless, we find that the corresponding value of the Kontorovich–Lebedev transform (1.2) is $\int_0^1 K_{i\tau}(x)\,dx$. The latter integral is a continuous function with respect to τ and behaves as $O(e^{-\pi\tau/2})$ when $\tau \to +\infty$ (cf. [7]). Consequently, integral (1.21) is divergent for this function when $1 \leqslant p < 2$.

In order to establish inequality (1.19) we need the boundedness of the Kontorovich–Lebedev transform as an operator (see (1.15), (1.22))

$$K_{i\tau} : L_\infty(\mathbb{R}_+; x\,dx) \to L_\infty(\mathbb{R}_+; \tau \sinh \pi\tau\,d\tau).$$

This result is given by

Theorem 4. *The Kontorovich–Lebedev transformation* (1.2) *is a bounded operator* $L_\infty(\mathbb{R}_+; x\,dx) \to L_\infty(\mathbb{R}_+; \tau \sinh \pi\tau\,d\tau)$ *and its norm is equal to* $\frac{\pi}{2}$.

Proof. Taking into account inequality (1.10) for the modified Bessel function and definitions of norms (1.15), (1.22) we deduce from (1.2) the following estimate:

$$||K_{i\tau}[f]||_{L_\infty(\mathbb{R}_+;\tau \sinh \pi\tau\, d\tau)} \leqslant ||f||_{L_\infty(\mathbb{R}_+;x\,dx)} \int_0^\infty K_0(x)\,dx$$
$$= \frac{\pi}{2}||f||_{L_\infty(\mathbb{R}_+;x\,dx)}, \tag{3.2}$$

where the latter integral is equal to $\frac{\pi}{2}$ due to relation (2.16.2.1) from [4, vol. II]. Consequently, the Kontorovich–Lebedev transform is bounded from $L_\infty(\mathbb{R}_+; x\,dx)$ into the space $L_\infty(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$ and its norm (1.23) $||K_{i\tau}|| \leqslant \frac{\pi}{2}$. However, it attains its least value by taking $f(x) \equiv 1$. Indeed, by the same relation (2.16.2.1) from [4, vol. II] we get that the transform (1.2) is

$$g(\tau) = \int_0^\infty K_{i\tau}(x)\,dx = \frac{\pi}{2}\frac{1}{\cosh\left(\frac{\pi\tau}{2}\right)},$$

and clearly, $||g||_{L_\infty(\mathbb{R}_+;\tau \sinh \pi\tau\, d\tau)} = \frac{\pi}{2}$. Theorem 4 is proved. □

It is not difficult to find the norm of the Kontorovich–Lebedev operator when $p = 2$. In this case the Parseval equality (1.28) gives the value $\frac{\pi}{\sqrt{2}}$. When $2 < p < \infty$, the norm estimate is established by the Riesz–Kolmogorov-type theorem. So, finally we prove

Theorem 5. *Let* $2 \leqslant p \leqslant \infty$. *The Kontorovich–Lebedev transformation* (1.2) *is a bounded operator* $L_p(\mathbb{R}_+; x\,dx) \to L_p(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$. *Moreover, inequality* (1.19) *holds true and the norm* (1.23) $||K_{i\tau}|| \leqslant \frac{\pi}{2^{1-\frac{1}{p}}}$.

Proof. In fact, by the Plancherel-type theorem for the Kontorovich–Lebedev transformation [7] we conclude that integral operator $K_{i\tau}$ is of type $(2, 2)$. Meantime, Theorem 4 states that this operator is of type (∞, ∞). Hence by the Riesz–Thorin convexity theorem [2] the Kontorovich–Lebedev operator (1.2) is of type (p, p), i.e. maps the space $L_p(\mathbb{R}_+; x\,dx)$ into $L_p(\mathbb{R}_+; \tau \sinh \pi\tau\, d\tau)$, where $p^{-1} = \frac{\theta}{2},\ 0 \leqslant \theta \leqslant 1$. This means that $2 \leqslant p \leqslant \infty$ and we find

$$||K_{i\tau}[f]||_{L_p(\mathbb{R}_+;\tau \sinh \pi\tau\, d\tau)} \leqslant \left(\frac{\pi}{\sqrt{2}}\right)^{\theta}\left(\frac{\pi}{2}\right)^{1-\theta}||f||_{L_p(\mathbb{R}_+;x\,dx)}$$
$$= \left(\frac{\pi}{\sqrt{2}}\right)^{2/p}\left(\frac{\pi}{2}\right)^{1-2/p}$$
$$\times ||f||_{L_p(\mathbb{R}_+;x\,dx)} = \frac{\pi}{2^{1-\frac{1}{p}}}||f||_{L_p(\mathbb{R}_+;x\,dx)}. \tag{3.3}$$

Thus we easily get inequality (1.19). Furthermore, from (3.3) and (1.23) it follows that $||K_{i\tau}|| \leqslant \frac{\pi}{2^{1-\frac{1}{p}}}$. Theorem 5 is proved. □

Remark 2. The problem of the inversion formula for the Kontorovich–Lebedev transformation (1.2) is considered in [8]. When $p = 2$ it is given by relations (1.29), (1.30). But if $2 < p < \infty$ the corresponding relation is a particular case of the main theorem in [8] and reciprocally to (1.2) we obtain

$$f(x) = \lim_{\varepsilon \to 0+} \frac{2}{x\pi^2} \int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) K_{i\tau}(x) K_{i\tau}[f]\, d\tau, \quad x > 0,$$

where the limit is taken with respect to the norm (1.14) or exists almost for all $x > 0$.

Our final conjecture states that the norm (1.23) of the Kontorovich–Lebedev operator is equal to $\frac{\pi}{2^{1-\frac{1}{p}}}$. However, it is still an open question for $2 < p < \infty$.

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[2] R.E. Edwards, Fourier Series II, Holt Rinehart and Winston, New York, 1967.
[3] S.K. Pichorides, On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov, Studia Math. 44 (1972) 165–179.
[4] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. I: Elementary Functions, vol. II: Special Functions, Gordon & Breach, New York, London, 1986.
[5] I.N. Sneddon, The Use of Integral Transforms, McGraw-Hill, New York, 1972.
[6] S.B. Yakubovich, On the convolution for the Kontorovich–Lebedev transform and its applications to integral equations, Dokl. Akad. Nauk BSSR 31 (2) (1987) 101–103 (in Russian).
[7] S.B. Yakubovich, Index Transforms, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1996.
[8] S.B. Yakubovich, On the Kontorovich–Lebedev transformation, J. Integral Equations Appl. 15 (1) (2003) 95–112.
[9] S.B. Yakubovich, Integral transforms of the Kontorovich–Lebedev convolution type, Collect. Math. 54 (2) (2003) 99–110.
[10] S.B. Yakubovich, Boundedness and inversion properties of certain convolution transforms, J. Korean Math. Soc. 40 (6) (2003) 999–1014.
[11] S.B. Yakubovich, Yu.F. Luchko, The Hypergeometric Approach to Integral Transforms and Convolutions (Kluwers Series in Mathematics and Applications vol. 287), Dordrecht, Boston, London, 1994.